Bayesian Estimation of the Parameters Half-Normal Distribution Using Lindley's Approximation

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Abstract— **In this paper, we address the problem of estimating the parameters of the Half-Normal distribution using various methods, including Maximum Likelihood Estimation, the Method of Moments, and Bayesian estimation under different prior distributions and loss functions. When closed-form Bayesian estimators are unavailable, Lindley's approximation is applied. Additionally, using Mathematica 12, a simulation study is conducted to illustrate numerical examples and evaluate the performance of the estimators.**

***Keywords*— distribution, Fisher information, Lindley's approximation.**

I. INTRODUCTION

The Half-Normal distribution HN (0, θ^2) is a probability distribution often used in statistical modeling. Its applications are diverse and can be found in various fields quality control, Environmental science, biological sciences, Psychology and social sciences and risk analysis assessment and management, the half-normal distribution is centered around zero and restricted to non-negative values, making it useful for data modeling where negative values are not possible but where the data still exhibit the characteristics of a normal distribution on the positive side. can be used to model the distribution of potential losses or risks associated with certain events or decisions (Jingchao et al., (2021). then the random variable follows from the following PDF:

$$f(x) = \sqrt{\frac{2}{\pi\theta^2}}\; e^{\frac{-x^2}{2\theta^2}} \quad , \; x \geq 0$$

and the cumulative distribution function CDF:

$$F(x) = P(X \leq x) = erf\left(\frac{x}{\theta\sqrt{2}}\right)$$

II. LINDLEY'S APPROXIMATION FORMULA FOR BAYESIAN ESTIMATORS

The basic idea in Lindley' s approach is to obtain Taylor series expansion of the function involved in posterior moment, assume that $L(\theta)$ be the log-likelihood of a random sample from X that has a PDF $f(x,\theta)$. Suppose that $\rho(\theta)$ is the logarithm of the joint prior distribution of θ. Let $\theta = \{\theta\_1, \theta\_2, \ldots\ldots, \theta\_m\}$, and $u(\theta)$ be a function of θ that is differentiable with respect to all its components.

$$E(u(\theta)|S) = \frac{\int u(\theta)e^{(\rho(\theta)+L(\theta))}d\theta}{\int e^{(\rho(\theta)+L(\theta))}\,d\theta}$$

The basic idea in Lindley's approach is to obtain Taylor series expansion of the function involved in posterior moment. His formula is given as follows:

$$E(u(\theta)|S) \approx \hat{g} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{r=1}^{m}(L_{ijk}\sigma_{ij}\sigma_{kr}u_r) + o\left(\frac{1}{n^2}\right)$$

Where:

$$i,j,k,r = 1,2,\ldots\ldots m.$$

$$u_i=\frac{\partial u}{\partial\theta_i}, u_{ij}=\frac{\partial^2 u}{\partial\theta_i\partial\theta_j}, L_{ijk}=\frac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_k}, \rho=\frac{\partial\rho}{\partial\theta_i}.$$

and σ_{ij} is the $(i,j)^{th}$element in the multiplicative inverse of the matrix $\{-L_{ij}\}$ This formula holds for large sample size n and under some regularity conditions, and it should be evaluated at the maximum likelihood estimates (MLE' s) of the involved parameters in θ.
In our case $m=2$, with $\theta_1=\beta$ and $\theta_2=\lambda$ and hence we need only the following derivatives of $L(\theta)$.

$$L_{20}=\frac{\partial^2 L}{\partial\theta_1{}^2}, L_{11}=\frac{\partial^2 L}{\partial\theta_1\partial\theta_2}, L_{02}=\frac{\partial^2 L}{\partial\theta_2{}^2},$$
$$L_{30}=\frac{\partial^3 L}{\partial\theta_1{}^3}, L_{12}=\frac{\partial^3 L}{\partial\theta_1\partial\theta_2{}^2}, L_{21}=\frac{\partial^3 L}{\partial\theta_1{}^2\partial\theta_2}, L_{03}=\frac{\partial^3 L}{\partial\theta_2{}^3}.$$

The empirical Fisher information is

$$\widehat{FI}=\begin{pmatrix}-\frac{\partial^2 L}{\partial\theta_1{}^2} & -\frac{\partial^2 L}{\partial\theta_1\partial\theta_2}\\ -\frac{\partial^2 L}{\partial\theta_1\partial\theta_2} & -\frac{\partial^2 L}{\partial\theta_2{}^2}\end{pmatrix}$$

It be should be noted here that the exact Fisher information is taken as the matrix of the expected values of the above matrix. The determinant of $\widehat{FI}$ can be given by:

$$\Delta=\left(\frac{\partial^2 L}{\partial\theta_1{}^2}\frac{\partial^2 L}{\partial\theta_2{}^2}-\left(\frac{\partial^2 L}{\partial\theta_1\partial\theta_2}\right)^2\right)$$

$$\widehat{FI}^2=\frac{1}{\Delta}\begin{pmatrix}-\frac{\partial^2 L}{\partial\theta_2{}^2} & \frac{\partial^2 L}{\partial\theta_1\partial\theta_2}\\ \frac{\partial^2 L}{\partial\theta_1\partial\theta_2} & -\frac{\partial^2 L}{\partial\theta_1{}^2}\end{pmatrix}$$

The σ_{ij} terms of the above inverse of Fisher information can be states as follows:

$$\sigma_{11}=-\frac{1}{\Delta}\frac{\partial^2 L}{\partial\theta_2{}^2},$$
$$\sigma_{12}=\sigma_{21}=\frac{1}{\Delta}\frac{\partial^2 L}{\partial\theta_1\partial\theta_2},$$
$$\sigma_{22}=-\frac{1}{\Delta}\frac{\partial^2 L}{\partial\theta_1{}^2}.$$

III. Bayesian Estimation of the Parameters of the Gamma Distribution using Lindley's Approximation

we will discuss the gamma distribution Lindley. In this connection, we find that the PDF is of the form:

$$f(x)=\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}$$

Consequently, we can obtain the following states:
The joint prior distribution:
We will take $\alpha\sim R(c_1)$ and $\beta\sim HN(c_2)$

$$P_1(\alpha)=\frac{\alpha e^{-\frac{\alpha^2}{2c_1{}^2}}}{c_1{}^2}$$
$$P_2(\beta)=\frac{2c_2e^{-\frac{c_2{}^2\beta^2}{\pi}}}{\pi}$$

And α,β are independent
Then,

$$P(\alpha,\beta)=P_1P_2$$
$$P(\alpha,\beta)=\frac{2\alpha c_2e^{-\left(\frac{\alpha^2}{2c_1{}^2}+\frac{c_2{}^2\sigma^2}{\pi}\right)}}{c_1{}^2\pi}$$

Therefor we can have the log likelihood of a random sample of size n:

$$L=\sum_{j=1}^{n}\Big(-\alpha\,log(\sigma)-log\;(\Gamma(\alpha))-log(x_j)+\alpha\,log(x_j)-\frac{x_j}{\sigma}$$

Which implies required derivatives of L:

$$L_{03}=\sum_{j=1}^{n}\left(-\frac{2\alpha}{\beta^3}+\frac{6x_j}{\beta^4}\right),$$
$$L_{30}=-n\,\psi(2,\alpha).$$
$$L_{12}=\frac{n}{\beta^2},$$
$$L_{21}=0.$$

And we can have the logarithm of the joint prior distribution:

$$\rho=-\frac{\alpha^2}{2c_1{}^2}-\frac{c_2{}^2\beta^2}{\pi}+log(2)-2\,log(c_1)+log(C_2)-log(\pi)+log(\alpha)$$

Derivative of ρ:

$$\rho_{10}=\frac{\partial\rho}{\partial\alpha}=\frac{\frac{e^{-\frac{\alpha^2}{2c_1{}^2}}}{c_1{}^2}-\frac{\alpha^2e^{-\frac{\alpha^2}{2c_1{}^2}}}{c_1{}^4}}{\frac{\alpha e^{-\frac{\alpha^2}{2c_1{}^2}}}{c_1{}^2}}.$$

$$\rho_{01}=\frac{\partial\rho}{\partial\beta}=-\frac{2\beta c_2{}^2}{\pi}.$$

We want to find the MLE:

$$\frac{\partial L}{\partial\alpha}=\sum_{j=1}^{n}(-\log(\beta)+\log(x_j)-\psi(\alpha)=0$$

$$\frac{\partial L}{\partial\beta}=\sum_{j=1}^{n}\left(-\frac{\alpha}{\beta}+\frac{x_j}{\beta^2}\right)=0$$

And,

$$\hat{\alpha}=\frac{\sum_{j=1}^{n}\log(x_j)}{n}$$

$$\hat{\beta}=\sqrt{\frac{\sum_{i=1}^{n}\log(x_i)^2}{n}-\frac{(\sum_{j=1}^{n}\log(x_j))^2}{n^2}}$$

Observed Fisher information matrix and its inverse $\widehat{FI}$ evaluated at $\hat{\alpha}$ and $\hat{\beta}$:

$$\sigma_{11} = -n\,\psi(1,\alpha)\,,$$
$$\sigma_{22} = \sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right),$$
$$\sigma_{12} = -\frac{n}{\beta}\,,$$
$$\sigma_{21} = -\frac{n}{\beta}\,.$$

And so, we obtain the following states:

$$\widehat{FI} = \begin{pmatrix} n\,\psi(1,\alpha) & \frac{n}{\beta} \\ \frac{n}{\beta} & -\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right) \end{pmatrix}$$

$$\Delta = \frac{-\left(n^2 + n\beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)}{\beta^2}$$

$$I\widehat{FI} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$$

Therefore,

$$\tau_{11} = \frac{\beta^2\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)}{n^2 + n\beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)}$$

$$\tau_{12} = \frac{\beta}{n + \beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)}$$

$$\tau_{21} = \frac{\beta}{n + \beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)}$$

$$\tau_{22} = -\frac{\beta^2\psi(1,\alpha)}{n + \beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)}$$

IV. Applied Lindley approximation for mean of the gamma distribution $g(\alpha,\beta) = \alpha\beta$

$$u_{12} = u_{21} = 1$$
$$u_{11} = u_{22} = 0$$
$$uu_1 = \beta$$
$$uu_2 = \alpha$$

1) Basic Components of the approximated Lindley estimator of the $g(\alpha,\beta)$:

$$A_{ij} = \left(\left(uu\tau_{ii} + uu_j\tau_{ij}\right)\tau_{ii}\right)$$

We obtain the following states:

$$A_{11} = \frac{2\beta^5\left(\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)^2}{\left(n^2 + n\beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)^2}$$

$$A_{12} = \frac{\beta^3\left(\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)\left(n\alpha + \beta^2\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)}{n^2\left(n + \beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)^2}$$

$$A_{22} = \frac{2\alpha\beta^4\psi(1,\alpha)^2}{\left(n + \beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)^2}$$
$$A_{21} = \frac{\beta^4\psi(1,\alpha)\left(\beta\psi(1,\alpha) - 1\right)}{\left(n + \beta^2\psi(1,\alpha)\sum_{j=1}^{n}\left(\frac{\alpha}{\beta^2} - \frac{2x_j}{\beta^3}\right)\right)^2}$$

And,

$$B_{ij} = \left(3uu_i\tau_{ii}\tau_{ij}\right) + uu_j\left(\tau_{ii}\tau_{jj} + 2\tau_{ij}{}^2\right)$$
$$B_{ij} = 0$$

And,

$$cc_{ij} = uu_i\tau_{ii} + uu_j\tau_{ji}$$
$$cc_{ij} = 0$$

2) We want to find the MLE of $g(\alpha,\beta)$:

$$\hat{g} = \frac{\left(\sum_{j=1}^{n} log\,(x_j)\right)\sqrt{\frac{n\sum_{i=1}^{n} log(x_i)^2}{n^2}}}{n}$$

3)The Bayes Lindley Approximated Estimate of $g(\alpha,\beta)$:

$$A_1 = \left(u_{10}\sigma_{11} + u_{01}\sigma_{12}\right)$$
$$A_2 = \left(u_{01}\sigma_{22} + u_{10}\sigma_{21}\right)$$

And,

$$B_1 = \left(l_{30}\sigma_{11} + l_{21}\sigma_{12} + l_{21}\sigma_{21} + l_{12}\sigma_{22}\right)$$
$$B_2 = \left(l_{21}\sigma_{11} + l_{12}\sigma_{12} + l_{12}\sigma_{21} + l_{03}\sigma_{22}\right)$$

And,

$$T_1 = \left(u_{20} + 2u_{10}\rho_{10}\right)\sigma_{11}$$
$$T_2 = \left(u_{11} + 2u_{01}\rho_{10}\right)\sigma_{21}$$
$$T_3 = \left(u_{11} + 2u_{10}\rho_{01}\right)\sigma_{12}$$
$$T_4 = \left(u_{02} + 2u_{01}\rho_{01}\right)\sigma_{22}$$

Therefore,

$$g_{BL} = \hat{g} + \frac{1}{2}\left(T_1 + T_2 + T_3 + T_4\right) + \frac{1}{2}\left(A_1B_1 + A_2B_2\right)$$

V. Conclusions

We have applied Bayesian techniques for estimating the parameters of the Half-Normal distribution the classical methods (maximum likelihood, method of moments) and the Bayesian method, where we used Lindley's approximation when we can't get Bayesian estimator in closed forms. We have further studied some characteristics of these estimators.

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