Distribution of Shannon Statistic from Weibull Sample

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Abstract— **In this paper, We calculates the exact quantiles of the Shannon entropy of the Weibull distribution's maximum likelihood estimator with those found in the asymptotic distribution of this statistic in order to comprehend the exact distribution of this statistic. However, it proposes a new version of that statistic and shows that, even for small samples, the modified statistic is considerably closer to normality than the one provided in the literature. Additionally, through the quantiles of the chi-square distribution, it offers formulas for quantiles of the precise distribution of both statistics. However, it derives some characteristics from the Shannon entropy of the Weibull distribution's maximum likelihood estimation, using Mathematica 12.**

Keywords— **Shannon entropy, MLE, Weibull distribution, Fisher information, Asymptotic normality, Extreme value distribution.**

I. Introduction

we focus on addressing these issues for a specific member of the exponential family, namely the Weibull distribution.

The critical points in a testing problem depend on the distribution of the test statistic. In some cases, when the exact distribution of the test statistic is unknown, it may be approximated using the central limit theorem. However, such an approximation can lead to critical points that result in decisions differing from those based on the exact distribution. This paper presents an example where this occurs, highlighting the need for more accurate approximations or modifications to the test statistic to reduce the risk of incorrect decisions.

II. Preliminary Results

Let $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy,\ \ \alpha > 0,\ and\ \psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ denote the gamma and digamma functions, respectively. This section provides some results about these two functions. These results will be used in the sequel

Lemma 1.

1) The n^{th} derivative of $\Gamma(k)$ is

$$\Gamma^{(n)}(k) = \int_0^\infty (\log(t))^n t^{k-1}e^{-t}dt\,, \qquad k > 0.$$

2) $\psi'(k) = \frac{\Gamma''(k)}{\Gamma(k)} - \psi^2(k).$

3) $\psi''(k) = \frac{\Gamma'''(k)}{\Gamma(k)} - 3\psi(k)\,\psi'(k) - \psi^3(k).$

4) $\psi'''(k) = \frac{\Gamma^{(4)}(k)}{\Gamma(k)} - 4\psi(k)\,\psi''(k) - 6\,\psi^2(k)\psi'(k) - 3\psi'^2(k) - \psi^4(k).$

Proof. Straight forward calculations lead to these results.

Corollary1. Let $Z: G(\alpha, 1)$ denote a gamma random variable with parameters$(\alpha, 1)$, then

1) $\psi(\alpha) = E(\log(Z))$
2) $\psi'(\alpha) = Var\,(\log(Z))$
3) $\psi''(\alpha)\ = E(\log\,(Z) - E(\log(Z))^3$
4) $\psi'''(\alpha) = E\big(\log(Z) - E(\log(Z))\big)^4 - 3\left(Var\big(log(Z)\big)\right)^2$

Proof. see [11]

Corollary 2. Let $S: G(\alpha, \beta)$ denote a gamma random variable with parameters (α, β),then

?

1) $E(log(S)) \ = \psi(\alpha) + log(\beta)$
2) $Var(log(S)) \ = \psi \ \ (\alpha)$
3) Skewness coefficient of $log(S)$ is $\frac{\psi''(\alpha)}{\sqrt{(\psi'(\alpha))^3}}$
4) Kurtosis coefficient of $\log(S)$ is $3 + \frac{\psi'''(\alpha)}{(\psi'(\alpha))^2}$

Proof. See [11]

III. MLE for Shannon entropy of Weibull distribution

Let T:Weibull(a, c) denote a Weibull random variable with mean $a$and b. Then Shannon entropy of this distribution is:

$$H(c) = log\left(\frac{c}{a}\right) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1).$$

The MLE for c based on r.s $\mathrm{T_1, T_2, \ldots, T_n}$ from $\mathrm{Weibull(a, b)}$ is $\hat{c} = \left(\frac{\sum_{i=1}^{n} t_i{}^{a}}{n}\right)^{\frac{1}{a}}$.

Hence, by the invariance property, the MLE for $H(\hat{c})$ is

$$\mathrm{H}(\hat{c}) = \log\left(\frac{\hat{c}}{a}\right) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1).$$

Lemma 2**.** Provides the properties of this estimator:

1) $\mathrm{E}(\mathrm{H}(\hat{c})) = \mathrm{H(c)} + \frac{1}{2}\left(\psi(0, \alpha n) - \log\left(\frac{\alpha n}{2}\right)\right)$, which is increasing in n.
2) $\mathrm{Var}(\mathrm{H}(\hat{c})) = (\psi'(\mathrm{a\,n}))$. $\forall\hat{c}$ which decreases in n to 0, but $\forall\hat{c}$,
$n\,Var(\mathrm{H}(\hat{c})) \to 1, as\ n \to \infty$.

Proof.

$$\begin{aligned}\mathrm{H}(\hat{c}) &= \log(\hat{c}) - \log(a) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1).\\ &= \log\left(\left(\frac{\sum_{i=1}^{n} t_i{}^{a}}{n}\right)^{\frac{1}{a}}\right) - \log(a) + 1 \\ &\quad - \left(\frac{a-1}{a}\right)\psi(0,1)\\ &= \log\left(\frac{n^{\frac{1}{a}}}{c}\left(\frac{\sum_{i=1}^{n} t_i{}^{a}}{n}\right)^{\frac{1}{a}}\right) - \log(a) + 1\\ &\quad - \left(\frac{a-1}{a}\right)\psi(0,1) - \log\left(\frac{c}{n^{\frac{1}{a}}}\right)\end{aligned}$$

Where
$\frac{n^{\frac{1}{a}}}{c}\left(\frac{\sum_{i=1}^{n} t_i{}^{a}}{n}\right)^{\frac{1}{a}} : G(a\,n, 1)$

$$\begin{aligned}E(H(\hat{c})) &= \psi(a\,n) - \log(a) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1)\\ &\quad + \log\left(\frac{c}{n^{\frac{1}{a}}}\right)\\ &= \psi(a\,n) + \log\left(\frac{c}{a}\right) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1) - \frac{1}{a}\log(n)\\ &= H(c) + \psi(a\,n) - \frac{1}{a}\log(n)\end{aligned}$$

Lemma 3. The estimator $\mathrm{H}(\hat{c})$ for $\mathrm{H(c)}$ is:

1) Biased, with bias equal to $\psi(a\,n) - \frac{1}{a}\log(n)$
2) Asymptotically unbiased.
3) Consistent.

IV. Confidence Interval for $H(\hat{c})$

Lemma 4. Consider $\underline{\mathrm{T}} = (\mathrm{T_1, T_2, \ldots, T_n})$as a r. s from a Weibull distribution variable with a and c. Then $\left(L(\underline{\mathrm{T}}), U(\underline{\mathrm{T}})\right)$. $(1-\alpha)100\%$ C.I for $\mathrm{H}(\hat{c})$ is

$$\begin{aligned}1 - \alpha &= \mathrm{P}\left(L(\underline{\mathrm{T}}) < \mathrm{H(c)} < U(\underline{\mathrm{T}})\right).\\ &= \mathrm{P}\left(L(\underline{\mathrm{T}}) < \mathrm{H}(\hat{c}) - \log\left(\sum_{i=1}^{n}\frac{t_i{}^{a}}{n}\right)^{\frac{1}{a}} + \log(\mathrm{a\,n}) < U(\underline{\mathrm{T}})\right)\\ &= \mathrm{P}\left(-U(\underline{\mathrm{T}}) < -\mathrm{H}(\hat{c}) + \log\left(\sum_{i=1}^{n}\frac{t_i{}^{a}}{n}\right)^{\frac{1}{a}} - \log(\mathrm{a\,n}) < -L(\underline{\mathrm{T}})\right)\\ &= \mathrm{P}\left(\mathrm{H}(\hat{c}) - U(\underline{\mathrm{T}}) < \log\left(\sum_{i=1}^{n}\frac{t_i{}^{a}}{n}\right)^{\frac{1}{a}} - \log(\mathrm{a\,n}) < \mathrm{H}(\hat{c}) - L(\underline{\mathrm{T}})\right)\\ &= \mathrm{P}\left(\mathrm{H}(\hat{c}) - U(\underline{\mathrm{T}}) + \log(\mathrm{a\,n}) < \log\left(\sum_{i=1}^{n}\frac{t_i{}^{a}}{n}\right)^{\frac{1}{a}} < \mathrm{H}(\hat{c}) - L(\underline{\mathrm{T}}) + \frac{1}{2}\log(\mathrm{a\,n})\right)\end{aligned}$$

$$= P\left(e^{(H(\hat{c})-U(\underline{T})+\log(a\,n))} < \left(\sum_{i=1}^{n} \frac{t_i{}^a}{n}\right)^{\frac{1}{a}} < e^{(H(\hat{c})-L(\underline{T})+\log(a\,n))} \right)$$

$$= P\left(e^{(H(\hat{c})-U(\underline{T})+\log(a\,n))} < \left(\sum_{i=1}^{n} \frac{t_i{}^a}{n}\right)^{\frac{1}{a}} < e^{(H(\hat{c})-L(\underline{T})+\log(a\,n))} \right)$$

$$= P\left(e^{(H(\hat{c})-U(\underline{T})+\log(a\,n))} < \frac{c}{n^{\frac{1}{a}}} \mathcal{X}^2{}_{[2\alpha n]} < e^{(H(\hat{c})-L(\underline{T})+\log(a\,n))} \right)$$

So,

$$e^{(H(\hat{c})-U(\underline{T})+\log(an))} = \mathcal{X}^2{}_{\left[\frac{\alpha}{2},2a\,n\right]}$$

$$e^{(H(\hat{c})-L(\underline{T})+\log(a\,n))} = \mathcal{X}^2{}_{\left[1-\frac{\alpha}{2},2a\,n\right]}$$

Where $\mathcal{X}^2{}_{[q\,,2\,a\,n]}$ is q^{th} percentile of $\mathcal{X}^2{}_{[2\,a\,n]}$.

Hence,

$$U(\underline{T}) = \frac{a\,n\,H(\hat{c})}{log\left(\mathcal{X}^2{}_{\left[\frac{\alpha}{2},2\alpha n\right]}\right)}.$$

$$L(\underline{T}) = \frac{a\,n\,H(\hat{c})}{log\left(\mathcal{X}^2{}_{\left[1-\frac{\alpha}{2},2\alpha n\right]}\right)}$$

V. Exact Distribution of $H(\hat{c})$

Lemma 5. The random variable

$W = \frac{\sqrt{n}(H(\hat{c})-H(c))}{\sqrt{\sigma^2(c)}}$ has an extreme value distribution of type one.

$$\sigma^2(c) = \frac{t^2}{I(c)} = \frac{1}{a^2}$$

$$t = \frac{\partial H(c)}{\partial c} = \frac{1}{c}.$$

Proof: $H(\hat{c}) - H(c) = \log(\hat{c}) - \log(c)$.

Based on the previous lemma 3., we can deduce the following results:

$$H(\hat{c}) - H(c) = \log\left(\sum_{i=1}^{n} \frac{t_i{}^a}{n}\right)^{\frac{1}{a}} - \log(c)$$

$$= \frac{1}{a}\log\left(\frac{1}{n}\sum_{i=1}^{n} t_i{}^a\right) - \log(c).$$

$$= \frac{1}{a}\log\left(\sum_{i=1}^{n}\left(\frac{t_i}{c}\right)^a\right) - \frac{1}{a}\log(n).$$

Hence,

$$W = a\sqrt{n}\left(\frac{1}{a}\log\left(\sum_{i=1}^{n}\left(\frac{t_i}{c}\right)^a\right) - \frac{1}{a}\log(n)\right).$$

$$= \sqrt{n}\log\left(\sum_{i=1}^{n}\left(\frac{t_i}{c}\right)^a\right) + \sqrt{n}\,\log\left(\frac{1}{n}\right)$$

Let $b = \sqrt{n}$, $d = -\sqrt{n}\,\log(n)$, $Y_i = \left(\frac{T_i}{c}\right)^a$ *and* $\zeta = \sum_{i=1}^{n} Y_i$.

Then by Lemma () and the fact that $Y_1, Y_2, \dots, Y_n$ are independent

$$\zeta = \sum_{i=1}^{n} Y_i \sim G(n,1)$$

Now, we should first observe that the pdf of ζ is given by:

$$f(\zeta) = \frac{\lambda^{n-1}e^{-\zeta}}{\Gamma(n)}, \zeta > 0.$$

Using transformation method we get the p.d.f of W as:

$$f(w) = \frac{e^{n\left(\frac{w-d}{b}\right)} e^{-e^{\left(\frac{w-d}{b}\right)}}}{b\Gamma(n)}.$$

Hence,

W has an extreme value distribution.

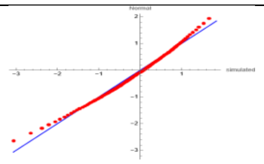
VI. Asymptotic Normality of Shannon Statistics

"Applying asymptotic normality result that is given by[12]and stated in section 1, it can be shown that for" Weibull distribution with parameters a, c.

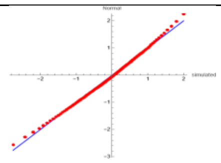
1) Fisher information $I(c) = \frac{a^2}{c^2}$.
2) $t = \frac{1}{c}$.
3) $\sigma^2(c) = \frac{1}{a^2}$.
4) $\sqrt{2n}\left(H(\hat{c}) - H(c)\right) \xrightarrow{L} N(0,1)\ as\ n \to \infty$

“To check the validity of asymptotic normality in case of small samples we have plotted the quantiles of the Weibull approximation vs. the exact quantiles of the exact distribution of Shannon statistic for $n = 5(5)\,35$. The obtained plots are reported in Figure (1)”.

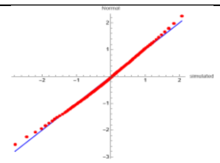
n =5



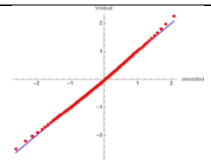
n =10



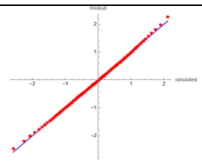
n =15



n =25



n =30



n =35

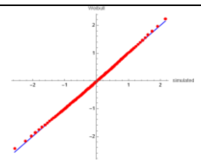


Figure 1.P-p- of Weibull Approximation vs. Exact Distribution of Standard Shannon, when $a = 2, c = 2,\ n_i = 1000, m = 100$.

Table 1**.** Quantiles of Shannon information based on r.s form Weibull(1,0.1), number of simulated samples $\mathrm{n_i} = 1000$.

n	0.025	0.05	0.1	0.25	0.75
1	-3.49773	-3.12956	-2.24419	-1.24198	0.379129
2	-5.13595	-4.06557	-3.23078	-1.75988	0.333111
3	-6.88649	-4.63844	-3.93793	-2.02719	0.636744
4	-6.747	-6.01098	-4.86927	-2.62061	0.539366
5	-7.88672	-6.4942	-5.5425	-2.66724	0.578223
6	-8.72676	-7.25782	-5.73616	-2.78728	0.923292
7	-9.59046	-8.17075	-6.09289	-2.89456	0.95058
8	-10.5046	-8.56333	-6.45385	-3.67359	1.20764
9	-10.2919	-9.24686	-6.23266	-4.03349	0.824387
10	-11.6347	-8.89788	-6.81595	-3.93522	0.923322
11	-12.4088	-9.50303	-7.1158	-4.17302	0.734093
12	-13.2938	-9.61139	-7.65258	-4.42443	1.06195
13	-12.6301	-10.1039	-8.15901	-4.3012	1.23198
14	-13.7465	-10.8113	-8.95753	-5.30567	1.18435
15	-14.4802	-10.9401	-8.75251	-4.85207	0.911955

Table 2. Quantiles of Shannon information based on r.s form Weibull(1,0.5), number of simulated samples $\mathrm{n_i} = 1000$

n	0.975	0.95	0.9	0.75	0.25
9	3.89853	3.08416	2.43844	1.03987	-3.68589
15	5.03515	3.80249	3.26946	1.3628	-4.78559
17	5.78553	4.55684	3.34657	1.4714	-4.72847
18	5.79682	4.99421	3.35576	1.58137	-4.99201
19	5.66143	4.86934	3.57784	1.39556	-5.47142
20	6.04764	5.0978	3.84562	1.34234	-6.0149
21	5.9802	4.94758	3.61414	1.53522	-5.89226
22	5.96842	4.95079	3.89865	1.50768	-6.12154
23	6.35897	5.21556	3.9891	1.60013	-5.83824
24	6.6977	5.5259	4.04784	1.82764	-5.84302
25	6.22568	5.44149	4.51537	2.19975	-5.72006
26	6.43276	5.81444	3.90238	1.78918	-6.28719
27	6.46122	5.51398	4.17183	2.00676	-6.77603
28	6.83157	5.83663	4.31743	1.82604	-7.03057
29	6.47561	6.03548	3.96761	1.87216	-6.4434
30	7.17187	5.80064	5.03413	1.62735	-6.99647

VII. CONCLUSIONS

The problem of determining the Shannon entropy statistic distribution using random samples from the Weibull distribution was the focus of this thesis. Since a number of statistical inference problems relied on the distribution of the statistics under investigation, the thesis is essential to solving the aforementioned problem. Determining the exact or asymptotic distribution of certain statistics is therefore particularly crucial. This thesis accomplishes several goals, such as determining the precise distribution of the entropy statistics under investigation using MLE derived from random samples. Furthermore, the goal of this thesis is to determine the asymptotic distribution of some of these statistics. Lastly, we want to create Mathematica codes that will implement the results on simulated data Reviewing entropy measures, calculating the MLE of the Shannon entropy measure as specified based on the targeted random distribution, determining some properties of the MLE estimates of entropy measures, determining the precise distribution of the statistics under study, determining the asymptotic distribution of the entropy statistics, and finally applying all of the results through simulation comprised the overall methodology used.

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