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3rd

The Role of Artificial Intelligence in Sustainable Development

DISTRIBUTION OF SHANNON STATISTIC FROM WEIBULL SAMPLE

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Introduction

we focus on addressing these issues for a specific member of the exponential family, namely the Weibull distribution.

The critical points in a testing problem depend on the distribution of the test statistic. In some cases, when the exact distribution of the test statistic is unknown, it may be approximated using the central limit theorem. However, such an approximation can lead to critical points that result in decisions differing from those based on the exact distribution. This paper presents an example where this occurs, highlighting the need for more accurate approximations or modifications to the test statistic to reduce the risk of incorrect decisions.

Preliminary

$$\boldsymbol{\Gamma(b) = \int_0^\infty t^{b-1}e^{-t}dt\ , \qquad b > 0}$$

denote the gamma and digamma functions, respectively. This section provides some results about these two functions. These results will be used in the sequel

Lemma 1.

1) The n^{th} derivative of $\Gamma(b)$ is

$$\Gamma^{(n)}(b) = \int_0^\infty (\log(t))^n t^{b-1} e^{-t} dt\,, \qquad b > 0.$$

1) $\psi'(b) = \frac{\Gamma''(b)}{\Gamma(b)} - \psi^2(b).$

2) $\psi''(b) = \frac{\Gamma'''(b)}{\Gamma(b)} - 3\psi(b)\,\psi'(b) - \psi^3(b).$

3) $\psi'''(b) = \frac{\Gamma^{(4)}(b)}{\Gamma(b)} - 4\psi(b)\,\psi''(b) - 6\,\psi^2(b)\psi'(b)$

$- 3\psi'^2(b) - \psi^4(b).$

Digamma and Polygamma function

The digamma function $\psi(b)$ is the logarithmic derivative of the gamma function given by $\psi(b) = \frac{\Gamma'(b)}{\Gamma(b)}$. For integer arguments, the digamma function satisfies the relation

$$\psi^{(n)}(b) = \frac{\mathrm{d}^n \psi(b)}{\mathrm{d}b^n}$$

Corollary1

Let $Z: G(\alpha, 1)$ denote a gamma random variable with parameters$(\alpha, 1)$, then

1) $\psi(\alpha) = E(log(Z))$
2) $\psi'(\alpha) = Var(log(Z))$
3) $\psi''(\alpha) = E(\log(Z) - E(log(Z))^3$
4) $\psi'''(\alpha) = E\left(log(Z) - E(log(Z))\right)^4 - 3\left(Var(log(Z))\right)^2$

Shannon Entropy of Weibull (a, c)

- If $T \sim \text{Weibull}(a, c)$, then the Shannon entropy of T is

$$\mathrm{H}(\mathrm{c}) = \log\left(\frac{c}{a}\right) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1).$$

- The MLE for $H(\hat{\mathrm{c}})$ is

$$\mathbf{H}(\hat{\mathbf{c}}) = \mathbf{log}\left(\frac{\hat{\boldsymbol{c}}}{\boldsymbol{a}}\right) + \mathbf{1} - \left(\frac{\boldsymbol{a}-\mathbf{1}}{\boldsymbol{a}}\right)\boldsymbol{\psi}(\mathbf{0},\mathbf{1})$$

If X~Weibull (a, c) , then the MLE of the parameter c is given as follow:

$$\hat{c} = \frac{\sum_{i=1}^{n} x_i{}^2}{n}$$

The Fisher information for Weibull distribution is

$$I(c) = \frac{a^2}{c^2}$$

Corollary 2

. Let $S: G(\alpha, \beta)$ denote a gamma random variable with parameters (α, β),then

1) $E(log(S)) = \psi(\alpha) + log(\beta)$
2) $Var(log(S)) = \psi(\alpha)$
3) Skewness coefficient of $log(S)$ is $\frac{\psi''(\alpha)}{\sqrt{(\psi'(\alpha))^3}}$
4) Kurtosis coefficient of $\log(S)$ is $3 + \frac{\psi'''(\alpha)}{(\psi'(\alpha))^2}$

Lemma 2

The properties of this estimator:

1) $E\left(H(\hat{c})\right) = H(c) + \frac{1}{2}\left(\psi(0, \alpha n) - \log\left(\frac{\alpha n}{2}\right)\right)$, which is increasing in n

2) $Var\left(H(\hat{c})\right) = \left(\psi'(a\, n)\right)$. $\forall \hat{c}$ which decreases in n to 0, but $\forall \hat{c}$, $n\, Var\left(H(\hat{c})\right) \to 1$, $as\ n \to \infty$.

Proof.

$$\mathrm{H}(\hat{\mathrm{c}}) = \log(\hat{c}) - \log(a) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1).$$

$$= \log\left(\left(\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} t_{\mathrm{i}}{}^{\mathrm{a}}}{\mathrm{n}}\right)^{\frac{1}{a}}\right) - \log(a) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1)$$

$$= \log\left(\frac{n^{\frac{1}{a}}}{c}\left(\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} t_{\mathrm{i}}{}^{\mathrm{a}}}{\mathrm{n}}\right)^{\frac{1}{a}}\right) - \log(a) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1) - \log\left(\frac{c}{n^{\frac{1}{a}}}\right)$$

Where $\frac{n^{\frac{1}{a}}}{c}\left(\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} t_{\mathrm{i}}{}^{\mathrm{a}}}{\mathrm{n}}\right)^{\frac{1}{a}} : G(a\,n\,,1)$

$$E\left(H(\hat{c})\right) = \psi(a\,n) - \log(a) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1) + \log\left(\frac{c}{n^{\frac{1}{a}}}\right)$$

$$= \psi(a\,n) + \log\left(\frac{c}{a}\right) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1) - \frac{1}{a}\log(n)$$

$$= H(c) + \psi(a\,n) - \frac{1}{a}\log(n)$$

Where $\frac{n^{\frac{1}{a}}}{c}\left(\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} t_{\mathrm{i}}{}^{\mathrm{a}}}{\mathrm{n}}\right)^{\frac{1}{a}} : G(a\,n\,,1)$

$$E\left(H(\hat{c})\right) = \psi(a\,n) - \log(a) + 1 - \left(\frac{a-1}{a}\right)\psi(0,1) + \log\left(\frac{c}{n^{\frac{1}{a}}}\right)$$

Lemma 3

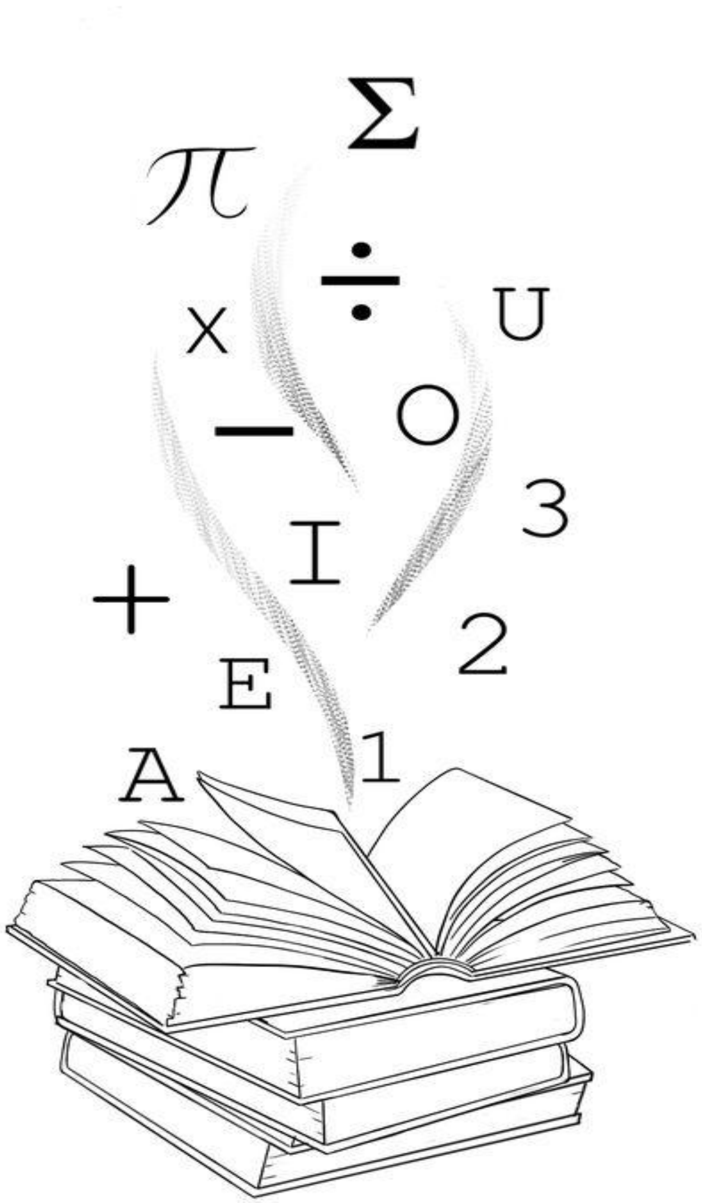
THE ESTIMATOR $H(\hat{c})$ FOR $H(c)$ IS:

1)Biased, with bias equal to

$$\psi(a\,n) - \frac{1}{a}\log(n)$$

1)Asymptotically unbiased.

2)Consistent.



Lemma 4

CONFIDENCE INTERVAL FOR H($\hat{c}$)

Consider $\underline{T} = (T_1, T_2, \ldots, T_n)$ as a random sample from a Weibull distribution variable with a and c.

Then $\left(L(\underline{T}), U(\underline{T})\right)$ be an exact $(1-\alpha)100\%$ confidence interval for H($\hat{c}$) is

$$U(\underline{T}) = \frac{\mathrm{a\ n\ H}(\hat{c})}{log\left(\mathcal{X}^2\left[\frac{\alpha}{2}, 2\alpha n\right]\right)}$$

$$L(\underline{T}) = \frac{\mathrm{a\ n\ H}(\hat{c})}{log\left(\mathcal{X}^2\left[1-\frac{\alpha}{2}, 2\alpha n\right]\right)}$$

Exact Distribution of $\mathrm{H}(\hat{\mathrm{c}})$

Lemma 5

The random variable $\mathrm{W} = \frac{\sqrt{\mathrm{n}}(\mathrm{H}(\hat{\mathrm{c}}) - \mathrm{H}(c))}{\sqrt{\sigma^2(\mathrm{c})}}$ has an extreme value distribution of type one.

where $\sigma^2(\mathrm{c}) = \frac{\mathrm{t}^2}{\mathrm{I}(\mathrm{c})} = \frac{1}{a^2}$;

$$\mathrm{t} = \frac{\partial \mathrm{H}(\mathrm{c})}{\partial \mathrm{c}} = \frac{1}{c}.$$

$$\boldsymbol{f(w) = \frac{e^{\mathrm{n}\left(\frac{w-d}{b}\right)} e^{-e^{\left(\frac{w-d}{b}\right)}}}{b\Gamma(n)}}$$

Proof: $H(\hat{c}) - H(c) = \log(\hat{c}) - \log(c)$.

Based on the previous lemma 3., we can deduce the

following results:

,

$$H(\hat{c}) - H(c) = \log\left(\sum_{i=1}^{n} \frac{t_i^{a}}{n}\right)^{\frac{1}{a}} - \log(c)$$

$$= \frac{1}{a}\log\left(\frac{1}{n}\sum_{i=1}^{n} t_i^{a}\right) - \log(c).$$

$$= \frac{1}{a}\log\left(\sum_{i=1}^{n}\left(\frac{t_i}{c}\right)^{a}\right) - \frac{1}{a}\log(n).$$

Hence,

$$W = a\sqrt{n}\left(\frac{1}{a}\log\left(\sum_{i=1}^{n}\left(\frac{t_i}{c}\right)^{a}\right) - \frac{1}{a}\log(n)\right).$$

$$= \sqrt{n}\log\left(\sum_{i=1}^{n}\left(\frac{t_i}{c}\right)^{a}\right) + \sqrt{n}\,\log\left(\frac{1}{n}\right)$$

Let $b = \sqrt{n}$, $d = -\sqrt{n}\,\log(n)$, $Y_i = \left(\frac{T_i}{c}\right)^{a}$ *and* $\zeta = \sum_{i=1}^{n} Y_i$.

Then by Lemma () and the fact that $Y_1, Y_2, \dots, Y_n$ are independent

$$\zeta = \sum_{i=1}^{n} Y_i \sim G(n, 1)$$

Now, we should first observe that the pdf of ζ is given by:

$$f(\zeta) = \frac{\lambda^{n-1} e^{-\zeta}}{\Gamma(n)}, \zeta > 0.$$

Using transformation method we get the p.d.f of W as:

$$\boldsymbol{f(w) = \frac{e^{n\left(\frac{w-d}{b}\right)} e^{-e^{\left(\frac{w-d}{b}\right)}}}{b\Gamma(n)}}$$

ASYMPTOTIC NORMALITY OF SHANNON STATISTICS

"Applying asymptotic normality result that is given by {12}and stated in section 1, it can be shown that for" Weibull distribution with parameters a, c.

1) Fisher information $\mathrm{I}(c) = \frac{a^2}{c^2}$.
2) $t = \frac{1}{c}$.
3) $\sigma^2(c) = \frac{1}{a^2}$.
4) $\sqrt{2n}\big(H(\hat{c}) - H(c)\big) \xrightarrow{L} N(0,1) \text{ as } n \to \infty$

Table 2. Quantiles of Shannon information based on r.s form Weibull(1,0.5), number of simulated samples $n_i = 1000$

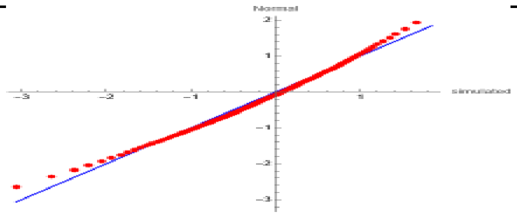
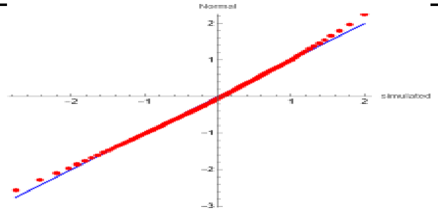
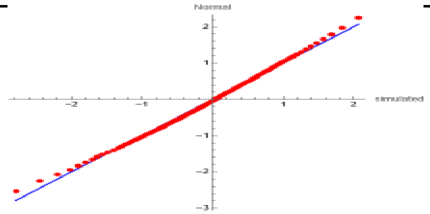
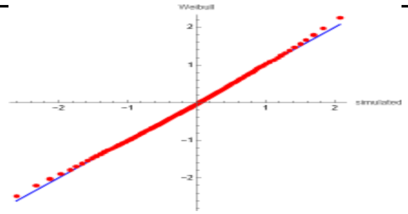
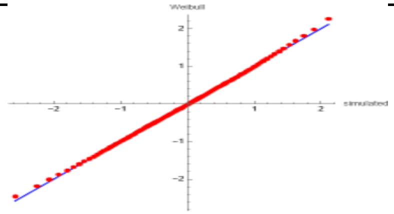
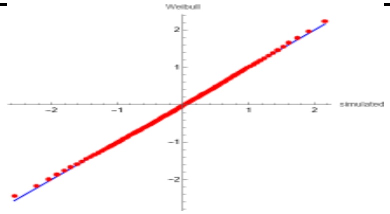
$n = 5$	$n = 10$
	
$n = 15$	$n = 25$
	
$n = 30$	$n = 35$
	

Figure 1. P-p- of Weibull Approximation vs. Exact Distribution of Standard Shannon, when $a = 2, c = 2,\ n_i = 1000, m = 100$.

Percentiles of Shannon entropy statistic based on sample form Weibull(1,0.1), number of simulated samples ni=1000.

n	0.025	0.05	0.75	0.9
1	-3.49773	-3.12956	0.379129	0.81179
2	-5.13595	-4.06557	0.333111	1.10463
3	-6.88649	-4.63844	0.636744	1.50666
4	-6.747	-6.01098	0.539366	1.62133
5	-7.88672	-6.4942	0.578223	1.80252
6	-8.72676	-7.25782	0.923292	2.0681
7	-9.59046	-8.17075	0.95058	2.16713
8	-10.5046	-8.56333	1.20764	2.53329
9	-10.2919	-9.24686	0.824387	2.47625
10	-11.6347	-8.89788	0.923322	2.54599
11	-12.4088	-9.50303	0.734093	2.83063
12	-13.2938	-9.61139	1.06195	2.70863
13	-12.6301	-10.1039	1.23198	3.03378
14	-13.7465	-10.8113	1.18435	3.36842
15	-14.4802	-10.9401	0.911955	3.06491
16	-14.999	-11.4545	1.51099	3.24217
17	-14.0176	-13.1508	1.50227	3.12001
18	-14.244	-12.529	1.14231	3.6564
19	-17.0768	-13.6419	1.18717	3.39837
20	-16.0513	-12.625	1.35285	3.60911
21	-16.3743	-13.763	1.32941	3.73983
22	-18.3349	-13.6207	1.34414	4.18056
23	-18.7083	-13.3138	1.18032	3.90741
24	-17.2152	-14.4043	1.51332	4.23329
25	-18.1768	-14.9519	1.51281	3.81978
26	-19.3653	-16.6843	1.86747	4.34485
27	-18.6553	-14.6016	1.66782	4.37522
28	-18.5482	-14.9983	1.64091	4.58942
29	-21.2056	-16.203	1.73088	4.5971
30	-19.9332	-15.9087	1.6958	4.77089

THANK YOU